The Mathematics of Poker-like Games

Maanas Belambe*
belambe2@illinois.edu

Aryan Gupta*
aryang9@illinois.edu

Adam Raizer*
araizer2@illinois.edu

Mihir Tandon*
mihirt2@illinois.edu

May 17, 2025

1 Introduction

The nuances of Poker forces players to leverage intuition on both their own hand as well as their opponent's in order to maximize their profit. To play Poker, one must simply have a deck of cards and an understanding of the relationship between pairs, straights, and flushes. However, to win [consistently], players must leverage an understanding of probabilities, combinatorics, and expected value. It is in this venue we direct our research, studying the relationship between strategies and equilibria for Bayesian games.

Our specific work derives its inspiration from a simplified model of Poker developed by Duke Economist David McAdams, which he called the "World's Simplest Poker". The two-player game operates as follows:

- **'Pre-flop':** The game features a symmetric blinds system. At the beginning of each round, both players, A and B, pay an "ante" of $1.
- **Dealing:** Both players are then simultaneously dealt a card represented by a random value in the interval $[0, 1]$. c_A denotes Player A's card value and c_B denotes Player B's card value.
- **Betting:** Once both players have seen their card, they can decide to either *bet* (where they must each contribute an additional $1 to the pot) or *fold*. Should the decisions of both players match, the game proceeds to a Showdown. In the case one player bets and the other folds, the betting player automatically wins the round.
- **Showdown:** Both cards are revealed. The player with the higher card value wins the showdown and wins the pot.

*Equal contribution.

From here, our group's work deviates into two unique analogues. The first is that of the continuous setting, where the McAdams Poker model remains unchanged. The second is the discrete setting, in which we replace the possible card values from a real-valued number in the range [0, 1] to an element in the discrete set $\{1, 2, 3...n\}$, where n is an arbitrary positive integer denoting the number of cards in the deck. For reference, we have categorized each round scenario and outcome in the following table:

Action	**Result**	**Profits**
Bet, Bet	$c_A > c_B$	$(2, -2)$
Bet, Bet	$c_A < c_B$	$(-2, 2)$
Fold, Fold	$c_A > c_B$	$(1, -1)$
Fold, Fold	$c_A < c_B$	$(-1, 1)$
Bet, Fold	Player A Wins	$(1, -1)$
Fold, Bet	Player B Wins	$(-1, 1)$

2 Strategies

To direct our work, we found it best to explicitly define different strategies that mimic human nature to help us understand the effects of one's playing style on their expected profit. The strategies are as follows:

- **Pure Cutoff Strategies:** A player bets if and only if their card $c_x \geq k_x$, where k_x is some arbitrary real-valued number in domain $[0, 1]$ (continuous) or item in the set $\{1, 2, 3...n\}$ (discrete), and folds otherwise.

- **Cutoff-Bluffing Strategies:** A player bets *with certainty* if their card $c_x \geq k_x$, or with probability p if $c_x < k_x$, and folds otherwise.

- **Pure Betting Strategies:** Exclusive to the discrete setting; A player bets if and only if their card $c_x \in \mathcal{B}_x$, where $\mathcal{B}_x$ is their discrete betting set, a subset of the deck, and folds otherwise.

- **Discrete General Strategies:** Exclusive to the discrete setting; A player's most general strategy can be expressed as a vector of probabilities,

$$\vec{p} = (p_1, p_2, p_3, \ldots, p_n),$$

  such that a player bets on card c_i with probability p_i.

3 Motivations

To guide our study, we sought to answer some defining questions. For the continuous setting,

- **Optimal Strategy:** What is the "optimal" strategy in this game? More specifically, on which cards c_x should the player bet in order to maximize their profit?
- **Best Response Strategy:** Given a strategy by Player A, what is the best response strategy for Player B? Can either player gain an advantage on the other by exploiting the sequential nature of the game?
- **Iterated Games:** What are the optimal strategies if the game is iterated either finitely or infinitely many times? If the number of rounds played is known in advance, does this change the "optimal" strategies?

And for the discrete setting,

- **Optimal Strategy:** What is the "optimal" strategy in this game? More specifically, given Player B uses a strategy with betting set $\mathcal{B}_B$, what is Player A's optimal response?
- **Bluffing:** If Player B uses some general strategy with probability vector $\vec{p}_B$, what is Player A's best response?
- **Nash Equilbrium:** In both the pure and bluffing strategies, does there exist a Nash Equilibrium? If so, is it unique?

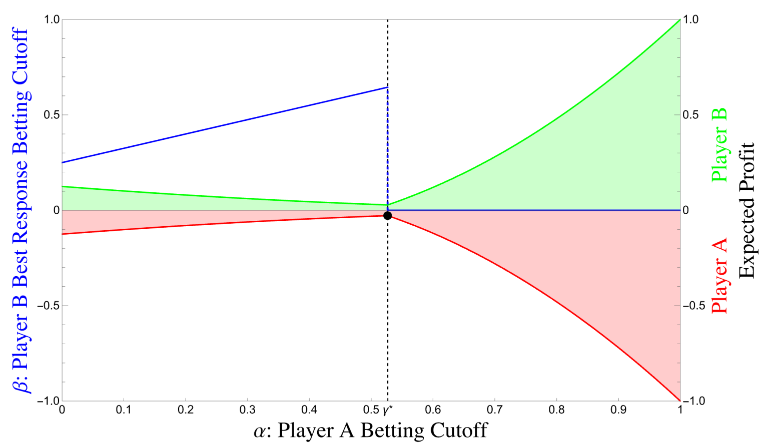
4 Results

4.1 Continuous Subgroup

For the continuous subgroup, we found the following results:

4.1.1 Best Response for Deterministic Cutoff Strategy

The natural first step for our analysis of this poker model is to determine how a certain player should respond based on their opponents strategy. We determined the following results in the analysis for the immediate best response strategy:



Theorem 1: Assume Player A and Player B employ deterministic strategies with betting cutoffs α and β, respectively. Then:

- The optimal choice of β, as a function of α, is:

$$\beta = \begin{cases} \frac{3\alpha+1}{4} & \text{if } \alpha < \gamma^*, \\ 0 & \text{if } \alpha > \gamma^*, \end{cases}$$

  where $\gamma^* = (3+2\sqrt{6})/15 \approx 0.527$. If $\alpha = \gamma^*$, there exist two equally optimal strategies of $\beta = (4+\sqrt{6})/10$ and $\beta = 0$.

- With this choice of β, the expected profit of Player B is:

$$P_B(\alpha) = \begin{cases} \frac{(\alpha-1)^2}{8} & \text{if } \alpha \leq \gamma^*, \\ 2\alpha^2 - \alpha & \text{if } \alpha \geq \gamma^*, \end{cases}$$

  which is strictly positive regardless of the value of α. The expected loss of Player A is $P_A(\alpha) = -P_B(\alpha)$.

- Assuming Player B plays the optimally as described above, Player A can minimize their expected loss by picking the strategy with betting cutoff $\alpha = \gamma^* \approx 0.527$.

4.1.2 Infinite Iteration of Best Response

Next, we wanted to explore what happens if we are to indefinitely iterate the best response strategies found above. We discovered the following behavior in our analysis of iterated best responses:

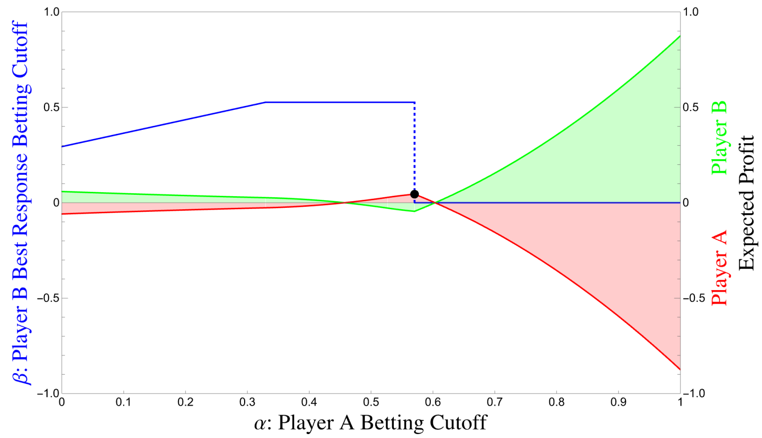
Theorem 2: Consider an infinite sequence of games where each player employs the best deterministic response to the opponent's strategy in the previous round. Then:

- The game always ends in the loop shown above consisting of the deterministic strategies with betting cutoffs of $0, \frac{1}{4}, \frac{7}{16}, \frac{37}{64}$.

- The player who first plays a strategy with a betting cutoff of $\frac{1}{4}$ will have a long-term positive expected profit and the player who first plays a strategy with a betting cutoff of 0 will have a long-term negative expected profit.

4.1.3 Finite Iteration of Best Response

Then, we wanted to see if anything changes if we are to limit the number of iterations to a finite number. In this scenario, we are assuming that both players know how many rounds are being played.



Theorem 3: Consider a game with two iterations where Player A employs an initial strategy with a betting cutoff α and Player B responds optimally with a with betting cutoff of β. If Player B considers that Player A will respond optimally to β, the optimal choice of β as a function of α is:

$$\beta = \begin{cases} \frac{12\alpha+5}{17} & \text{if } \alpha \leq \frac{17\gamma^*-5}{12}, \\ \gamma^* & \text{if } \frac{17\gamma^*-5}{12} \leq \alpha \leq \frac{14\sqrt{6}+171}{360}, \\ 0 & \text{if } \frac{14\sqrt{6}+171}{360} < \alpha. \end{cases}$$

With this choice of β, Player B has a positive expected profit unless α is in the interval $\approx (0.457, 0.604)$, in which case Player A has a positive expected profit. Player A's profit is maximized at $\alpha = (14\sqrt{6} + 171)/360 \approx 0.570$.

4.1.4 Best Response for Cutoff Strategy with Bluffing

Finally, we considered the possibility of bluffing as it is a key component of a real poker game. Based on our analysis of cutoff strategies with bluffing, we discovered the following results:

Theorem (McAdams): Among cutoff strategies with bluffing, the strategy pair $(1/2, 1/3)$ is a Nash Equilibrium, in the sense that neither player can gain an advantage by deviating from this strategy.

Theorem 4: The best response to a strategy (c, p) different than $(1/2, 1/3)$ is a **deterministic cutoff strategy** of the form $(c^*, 0)$ with an appropriate value of c^*. With this strategy, the expected profit of Player of B is strictly positive.

Theorem 5: The best response to a random play strategy (i.e. $(1, p)$) is to always bet (i.e. $(0, p)$).

4.1.5 Future Directions

In the future, we plan to expand on our work and explore the following:

- Among deterministic strategies, are cutoff strategies superior to all other deterministic strategies? (We proved that this is true for strategies where you bet if and only if your card value is in an interval.)
- Generalize Theorem 3 to iterated games involving more than two rounds. In other words, if both players know the number of rounds in advance and expect their opponent to respond optimally, what are the best strategies for each player? Can the player who moves last ensure a positive expected profit?
- Generalize the analysis to a game where the ante and bet amounts are general parameters (a, b).
- Generalize the analysis to a game where the betting player can determine how much he/she wants to bet within some range.

4.2 Discrete Subgroup

For the discrete subgroup, we found the following results:

4.2.1 Expected Payout for Cutoff Strategies

First a closed-form solution for the expected payout of a player given cutoff strategies. We found the relationship

$$\mathbb{E}_A(k_A, k_B) = \begin{cases} \frac{1}{n(n-1)}(3k_Ak_B - k_B^2 - 2k_A^2 + 2k_A - 2k_B + nk_A - nk_B), & k_A > k_B \\ \frac{1}{n(n-1)}(-3k_Ak_B + k_A^2 + 2k_B^2 - 2k_B + 2k_A - nk_B + nk_A), & k_A < k_B \\ 0, & else \end{cases}$$

depicted in Figure 1. Notice that, by the nature of the lattice, there exists **no global minimum** in $\mathbb{E}_A$.

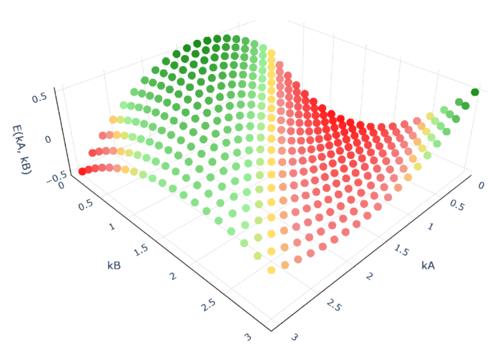


Figure 1: $\mathbb{E}_A(k_A, k_B)$ when $1 \leq k_A, k_B \leq n$

To gain a better intuition for the relationship between Player A's expected profit as a function of cutoff strategies, we can study the behavior in two-dimensions, By varying k_A and fixing k_B.

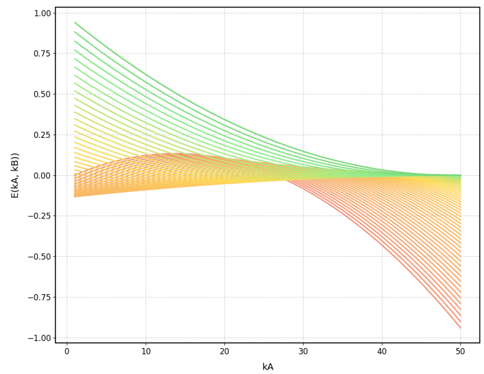


Figure 2: $\mathbb{E}_A(k_A, k_B)$ with constant k_B and $1 \leq k_A \leq n$

We see that as k_A increases with a low k_B, $\mathbb{E}_A$ decreases. Conversely, as k_A increases with a high k_B, $\mathbb{E}_A$ approaches zero. Thus, we can build the following intuitions regarding strategies which may serve as a foundation for optimal strategies:

- **Opponent plays safe**: When your opponent bets conservatively ($k_B \approx n$), your best move would be to play riskier ($k_A = 0$).
- **Opponent plays risky**: When your opponent bets riskily ($k_B \approx 0$), your best move would be to be slightly more conservative ($k_A \approx \frac{1}{4}n$).

Remark. The exact optimal k_A when the opponent plays risky is given by

$$k_A = \frac{2 + n + 3k_B}{4}.$$

We can perform case-by-case analysis for $2 + n + 3k_B \pmod 4$. However, it should be noted that

- **Cutoff Strategies are non-dominant**: As can be seen in Table 1 (coming soon), $\forall k_x, \exists \mathcal{B}_x$ such that $\mathbb{E}_A(\mathcal{B}_x) \geq \mathbb{E}_A(k_x)$.

As such, we can now direct our efforts to betting set and discrete general strategies in our search for dominant strategies and Nash Equilibria.

4.2.2 Iterated Strategies

By analyzing payouts for all possible pairs $\mathcal{B}_A, \mathcal{B}_B$ of betting sets (depicted in Table 1 for $n = 3$), Player A can find their best possible response $\mathcal{B}_A$ given $\mathcal{B}_B$.

1. **Usage**: Suppose Player B chooses $\mathcal{B}_B = \{2, 3\}$. Looking at the corresponding row, we see that $\mathcal{B}_A = \{3\}$ maximizes Player A's payout.
2. **Player B Counter-strategy**: As Player B aims to minimize Player A's profit, they would change to $\mathcal{B}_B = \{1, 3\}$.
3. **Player A Counter-counter-strategy**: Player A can then change to $\mathcal{B}_A = \{1, 2, 3\}$ in order to maximize their payout.

4. **Intransitivity**: Player B then chooses $\mathcal{B}_B = \{2, 3\}$, after which player A chooses $\mathcal{B}_A = \{3\}$, returning us to our starting point. This shows that the game is **intransitive**, as there exists a strategy cycle.

$\mathcal{B}_B \backslash \mathcal{B}_A$	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\emptyset$	0	4	2	0	6	4	2	6
$\{1\}$	-4	0	1	-1	5	3	3	8
$\{2\}$	-2	-1	0	1	1	2	3	4
$\{3\}$	0	1	-1	0	1	-1	-1	0
$\{1,2\}$	-6	-5	-1	0	0	1	5	6
$\{1,3\}$	-4	-3	-2	-1	-1	0	1	2
$\{2,3\}$	-2	-4	-3	1	-5	-1	0	-2
$\{1,2,3\}$	-6	-8	-4	0	-6	-2	2	0

Table 1: $\mathbb{E}_A$ for betting sets $\mathcal{B}_A, \mathcal{B}_B$ when $n = 3$.

4.2.3 Nash Equilibria & Dominance

As part of our game theoretic approach to the game, we search for the following:

- A **Nash Equilbrium**, where neither player has an incentive to change strategies in response to their opponent's current strategy.
- A **Dominant Strategy**, where one player always achieves their maximum payout by choosing a particular strategy.

We do this by laying the following groundwork:

- **Applying Nash's Theorem**: By *Nash's Theorem*, any game with a finite number of players and a finite number of pure strategies has at least one Nash Equilibrium.

Consider the case $n = 3$. We proved that, among the general strategies, the unique Nash Equilibrium occurs when both players adopt the strategy

$$(p_1, p_2, p_3) = (1/3, 1/3, 1).$$

Additionally, we proved that there is **no dominant strategy** for the discrete McAdams Model. In particular, the cutoff strategies are not dominant. For example, the optimal response to $\mathcal{B}_A = \{1\}$ is $\mathcal{B}_B = \{1, 3\}$, which is not a cutoff strategy.

4.2.4 Future Directions

In the future, we plan to continue our work and answer the following:

- **Deterministic Nash Equilibria**: We know, through computation, that there exists no deterministic Nash Equilibria for $n \leq 10$. Does there exist any deterministic Nash Equilibria for $n > 10$?
- **General Nash Equilibria**: Is there a pattern we can extrapolate to find General Strategy Nash Equilibria for any arbitrary n?
- **Evolving WSP**: How do the mechanics & optimal strategies change when considering multiple cards or rounds?

5 Links

1. https://github.com/aryan-cs/poker-like-games/tree/discrete-poker